

# A minimal coupling method for dissipative quantum systems

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## Abstract

Quantum dynamics of a general dissipative system investigated by its coupling to a Klein-Gordon type field as the environment by introducing a minimal coupling method. As an example, the quantum dynamics of a damped three dimensional harmonic oscillator investigated and some transition probabilities indicating the way energy flows between the subsystems obtained. The quantum dynamics of a dissipative two level system considered.

## 1 Introduction

In classical mechanics dissipation can be taken into account by introducing a velocity dependent damping term into the equation of motion. Such an approach is no longer possible in quantum mechanics where a time-independent Hamiltonian implies energy conservation and accordingly we can not find a unitary time evolution operator for both states and observable quantities consistently.

To investigate the quantum mechanical description of dissipating systems,

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there are some treatments, one can consider the interaction between two systems via an irreversible energy flow [1,2], or take a phenomenological treatment for a time dependent Hamiltonian which describes damped oscillations, here we can refer the interested reader to Caldirola-Kanai Hamiltonian for a damped harmonic oscillator [3].

$$H(t) = e^{-2\beta t} \frac{p^2}{2m} + e^{2\beta t} \frac{1}{2} m \omega^2 q^2. \quad (1)$$

There are significant difficulties about the quantum mechanical solutions of the Caldirola-Kanai Hamiltonian, for example quantizing such a Hamiltonian violates the uncertainty relations or canonical commutation rules and the uncertainty relations vanish as time tends to infinity.[4,5,6,7,8]

In 1931, Bateman [9] presented the mirror-image Hamiltonian which consists of two different oscillator, where one of them represents the main one-dimensional damped harmonic oscillator. Energy dissipated by the main oscillator completely will be absorbed by the other oscillator and thus the energy of the total system is conserved. Bateman Hamiltonian is given by

$$H = \frac{p\bar{p}}{m} + \frac{\beta}{2m}(\bar{x}\bar{p} - xp) + (k - \frac{\beta^2}{4m})x\bar{x}, \quad (2)$$

with the corresponding Lagrangian

$$L = m\dot{x}\dot{\bar{x}} + \frac{\beta}{2}(x\dot{\bar{x}} - \dot{x}\bar{x}) - kx\bar{x}, \quad (3)$$

canonical momenta for this dual system can be obtained from this Lagrangian as

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{\bar{x}} - \frac{\beta}{2}\bar{x}, \quad \bar{p} = \frac{\partial L}{\partial \dot{\bar{x}}} = m\dot{x} + \frac{\beta}{2}x, \quad (4)$$

dynamical variables  $x, p$  and  $\bar{p}, \bar{x}$  should satisfy the commutation relations

$$[x, p] = i, \quad [\bar{x}, \bar{p}] = i, \quad (5)$$

however the time-dependent uncertainty products obtained in this way, vanishes as time tends to infinity.[10]

Caldirola [3,11] developed a generalized quantum theory of a linear dissipative system in 1941 : equation of motion of a single particle subjected to a generalized non conservative force  $Q$  can be written as

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q} + Q(q), \quad (6)$$

where  $Q_r = -\beta(t) \sum a_{rj} \dot{q}_j$ , and  $a_{rj}$ 's are some constants, changing the variable  $t$  to  $t^*$ , using the following nonlinear transformation

$$t^* = \chi(t), \quad dt = \phi(t) dt^*, \quad \phi(t) = e^{\int_0^t \beta(t') dt'}, \quad (7)$$

together with the definitions

$$\dot{q}^* = \frac{dq}{dt^*}, \quad L^* = L(q, \dot{q}^*, t^*), \quad p^* = \frac{\partial L^*}{\partial \dot{q}^*}, \quad (8)$$

the Lagrangian equations, can be obtained from

$$\frac{d}{dt^*} \left( \frac{\partial L^*}{\partial \dot{q}^*} \right) - \frac{\partial L^*}{\partial q} = 0. \quad (9)$$

where  $H^* = \sum p^* \dot{q}^* - L^*$ . Canonical commutation rule and Schrodinger equation in this formalism are

$$[q, p^*] = i, \quad H^* \psi = i \frac{\partial \psi}{\partial t^*}, \quad (10)$$

but unfortunately uncertainty relations vanish as time goes to infinity.[10] Perhaps one of the effective approaches in quantum mechanics of dissipative systems is the idea of considering an environment coupled to the main system and doing calculations for the total system but at last for obtaining observables related to the main system, the environment degrees of freedom must be eliminated. The interested reader is referred to the Caldeira-Legget model [12,13]. In this model the dissipative system is coupled with an environment made by a collection of  $N$  harmonic oscillators with masses  $m_n$  and frequencies  $\omega_n$ , the interaction term in Hamiltonian is as follows

$$H' = -q \sum_{n=1}^N c_n x_n + q^2 \sum_{n=1}^N \frac{c_n^2}{2m_n \omega_n^2}, \quad (11)$$

where  $q$  and  $x_n$  denote coordinates of system and environment respectively and the constants  $c_n$  are called coupling constants.

The above coupling is not suitable for dissipative systems containing a dissipative term proportional to velocity. In fact with above coupling we can not obtain a Heisenberg equation like  $\ddot{q} + \omega^2 q + \beta \dot{q} = \xi(t)$ , for a damped harmonic oscillator, consistently. In this paper we generalize the Caldeira-Legget model to an environment with continuous degrees of freedom by a coupling

similar to the coupling between a charged particle and the electromagnetic field known as the minimal coupling. In section 2, the idea of a generalized minimal coupling method is introduced and in section 3 the quantum dynamics of a three dimensional damped harmonic oscillator is investigated. In section 4, we study quantum dynamics of a dissipative two level system. Finally in section 5, dynamics of quantum field of reservoir is investigated.

## 2 Quantum dynamics of a dissipative system

Quantum mechanics of a dissipative system can be investigated by introducing a reservoir or an environment that interacts with the system through a new kind of minimal coupling term. For this purpose let the damped system be a particle with mass  $m$  under an external potential  $v(\vec{x})$ . The total Hamiltonian, i.e., system plus environment, is

$$H = \frac{(\vec{p} - \vec{R})^2}{2m} + v(\vec{x}) + H_B, \quad (12)$$

where  $\vec{x}$  and  $\vec{p}$  are position and canonical conjugate momentum operators of the particle respectively and satisfy the canonical commutation rule

$$[\vec{x}, \vec{p}] = i, \quad (13)$$

$H_B$  is the reservoir Hamiltonian defined by

$$H_B(t) = \int_{-\infty}^{+\infty} d^3k \omega_{\vec{k}} b_{\vec{k}}^\dagger(t) b_{\vec{k}}(t), \quad \omega_{\vec{k}} = |\vec{k}|. \quad (14)$$

Annihilation and creation operators  $b_{\vec{k}}, b_{\vec{k}}^\dagger$ , in any instant of time, satisfy the following commutation relations

$$[b_{\vec{k}}(t), b_{\vec{k}'}^\dagger(t)] = \delta(\vec{k} - \vec{k}'), \quad (15)$$

and we will show later that reservoir is a Klein-Gordon type equation with a source term. Operator  $\vec{R}$  have the basic role in interaction between the system and reservoir and is defined by

$$\vec{R}(t) = \int_{-\infty}^{+\infty} d^3k [f(\omega_{\vec{k}}) b_{\vec{k}}(t) + f^*(\omega_{\vec{k}}) b_{\vec{k}}^\dagger(t)] \vec{k}, \quad (16)$$

let us call the function  $f(\omega_{\vec{k}})$ , the coupling function. It can be shown easily that Heisenberg equation for  $\vec{x}(t)$  and  $\vec{p}$  leads to

$$\begin{aligned}\dot{\vec{x}} &= i[H, \vec{x}] = \frac{\vec{p} - \vec{R}}{m}, \\ \dot{\vec{p}} &= i[H, \vec{p}] = -\vec{\nabla}v,\end{aligned}\tag{17}$$

where after omitting  $\vec{p}$ , gives the following equation for the damped system

$$m\ddot{\vec{x}} = -\vec{\nabla}v - \dot{\vec{R}}.\tag{18}$$

Using (15) the Heisenberg equation for  $b_{\vec{k}}$ , is

$$\dot{b}_{\vec{k}} = i[H, b_{\vec{k}}] = -i\omega_{\vec{k}}b_{\vec{k}} + if^*(\omega_{\vec{k}})\vec{k} \cdot \dot{\vec{x}},\tag{19}$$

with the following formal solution

$$b_{\vec{k}}(t) = b_{\vec{k}}(0)e^{-i\omega_{\vec{k}}t} + if^*(\omega_{\vec{k}})\vec{k} \cdot \int_0^t dt' e^{-i\omega_{\vec{k}}(t-t')} \dot{\vec{x}}(t'),\tag{20}$$

substituting  $b_{\vec{k}}(t)$  from (20) into (18), one obtains

$$\begin{aligned}m\ddot{\vec{x}} + \int_0^t dt' \dot{\vec{x}}(t')\gamma(t-t') &= -\vec{\nabla}v + \vec{\xi}(t) \\ \gamma(t) &= \frac{8\pi}{3} \int_0^\infty d\omega_{\vec{k}} |f(\omega_{\vec{k}})|^2 \omega_{\vec{k}}^5 \cos \omega_{\vec{k}}t \\ \vec{\xi}(t) &= i \int_{-\infty}^{+\infty} d^3k \omega_{\vec{k}} (f(\omega_{\vec{k}})b_{\vec{k}}(0)e^{-i\omega_{\vec{k}}t} - f^*(\omega_{\vec{k}})b_{\vec{k}}^\dagger(0)e^{i\omega_{\vec{k}}t})\vec{k}.\end{aligned}\tag{21}$$

It is clear that the expectation value of  $\vec{\xi}(t)$  in any eigenstate of  $H_B$ , is zero. For the following special choice of coupling function

$$f(\omega_{\vec{k}}) = \sqrt{\frac{3\beta}{4\pi^2\omega_{\vec{k}}^5}},\tag{22}$$

equation (21) takes the form

$$\begin{aligned}m\ddot{\vec{x}} + \beta\dot{\vec{x}} &= -\vec{\nabla}v + \vec{\xi}(t) \\ \vec{\xi}(t) &= i\sqrt{\frac{3\beta}{4\pi^2}} \int_{-\infty}^{+\infty} \frac{d^3k}{\sqrt{\omega_{\vec{k}}^3}} (b_{\vec{k}}(0)e^{-i\omega_{\vec{k}}t} - b_{\vec{k}}^\dagger(0)e^{i\omega_{\vec{k}}t}).\end{aligned}\tag{23}$$

In the following we investigate the quantum dynamics of a two level system and an isotropic three dimensional harmonic oscillator, both interacting with a reservoir, as prototypes of dissipative models.

### 3 Quantum dynamics of a three dimensional damped harmonic oscillator

#### 3.1 Quantum dynamics

For a three dimensional harmonic oscillator with mass  $m$  and frequency  $\omega$  we have  $v(\vec{x}) = \frac{1}{2}m^2\omega^2\vec{x}^2$  and therefore we can write (23) as

$$\ddot{\vec{x}} + \frac{\beta}{m}\dot{\vec{x}} + \omega^2\vec{x} = \frac{\vec{\xi}(t)}{m}, \quad (24)$$

with the following solution

$$\begin{aligned} \vec{x}(t) &= e^{-\frac{\beta t}{2m}}(\vec{A}e^{i\omega_1 t} + \vec{B}e^{-i\omega_1 t}) + \vec{M}(t), \\ \vec{M}(t) &= i \int_{-\infty}^{+\infty} d^3k \sqrt{\frac{3\beta}{4\pi^2 m^2 \omega_k^3}} \left[ \frac{b_k(0)}{\omega^2 - \omega_k^2 - i\frac{\beta}{m}\omega_k} e^{-i\omega_k t} - \frac{b_k^\dagger(0)}{\omega^2 - \omega_k^2 + i\frac{\beta}{m}\omega_k} e^{i\omega_k t} \right] \vec{k}, \end{aligned} \quad (25)$$

where  $\omega_1 = \sqrt{\omega^2 - \frac{\beta^2}{4m^2}}$ . Operators  $\vec{A}$  and  $\vec{B}$ , are specified by initial conditions

$$\begin{aligned} \vec{A} + \vec{B} &= \vec{x}(0) - \vec{M}(0), \\ \left(\frac{-\beta}{2m} + i\omega_1\right)\vec{A} + \left(\frac{-\beta}{2m} - i\omega_1\right)\vec{B} &= \dot{\vec{x}}(0) - \dot{\vec{M}}(0) \\ &= \frac{\vec{p}(0) - \vec{R}(0)}{m} - \dot{\vec{M}}(0), \end{aligned} \quad (26)$$

solving above equations and substituting  $\vec{A}$  and  $\vec{B}$  in (25), one obtains

$$\begin{aligned} \vec{x}(t) &= e^{-\frac{\beta t}{2m}} \left\{ \frac{\vec{p}(0)}{m\omega_1} \sin \omega_1 t + \vec{x}(0) \cos \omega_1 t + \frac{\beta}{2m\omega_1} \vec{x}(0) \sin \omega_1 t \right. \\ &\quad \left. - \frac{\vec{R}(0)}{m\omega_1} \sin \omega_1 t - \frac{\beta}{2m\omega_1} \vec{M}(0) \sin \omega_1 t - \vec{M}(0) \cos \omega_1 t - \frac{1}{\omega_1} \dot{\vec{M}}(0) \sin \omega_1 t \right\} + \vec{M}(t), \end{aligned} \quad (27)$$

also substituting  $\vec{x}(t)$  from (27) in (20) we can find a stable solution for  $b_{\vec{k}}(t)$  in  $t \rightarrow \infty$  as

$$\begin{aligned}
b_{\vec{k}}(t) = & b_{\vec{k}}(0)e^{-i\omega_{\vec{k}}t} - i\sqrt{\frac{3\beta}{4\pi^2\omega_{\vec{k}}^5}} \frac{e^{-i\omega_{\vec{k}}t}}{(\omega^2 - \omega_{\vec{k}}^2 - \frac{i\beta}{m}\omega_{\vec{k}})} \left\{ \omega^2 \vec{x}(0) + i\omega_{\vec{k}} \frac{\vec{p}(0) - \vec{R}(0)}{m} \right. \\
& \left. - \vec{M}(0)\omega^2 - i\omega_{\vec{k}} \dot{\vec{M}}(0) \right\} \\
& + \frac{3\beta i}{4\pi^2 m \sqrt{\omega_{\vec{k}}^5}} \int_{-\infty}^{+\infty} \frac{d^3 k'}{\sqrt{\omega_{\vec{k}'}^5}} \left\{ \frac{b_{\vec{k}'}(0)}{\omega^2 - \omega_{\vec{k}'}^2 - \frac{i\beta}{m}\omega_{\vec{k}'}} \frac{\sin \frac{(\omega_{\vec{k}} - \omega_{\vec{k}'})t}{2}}{\frac{(\omega_{\vec{k}} - \omega_{\vec{k}'})}{2}} e^{\frac{-i(\omega_{\vec{k}} + \omega_{\vec{k}'})t}{2}} \right. \\
& \left. + \frac{b_{\vec{k}'}^\dagger(0)}{\omega^2 - \omega_{\vec{k}'}^2 + \frac{i\beta}{m}\omega_{\vec{k}'}} \frac{\sin \frac{(\omega_{\vec{k}} + \omega_{\vec{k}'})t}{2}}{\frac{(\omega_{\vec{k}} + \omega_{\vec{k}'})}{2}} e^{\frac{i(\omega_{\vec{k}} - \omega_{\vec{k}'})t}{2}} \right\} \vec{k} \cdot \vec{k}', \tag{28}
\end{aligned}$$

now substituting  $b_{\vec{k}}(t)$  from (28) in (16) and using (17), one obtains  $\vec{p} = m\dot{\vec{x}} + \vec{R}$ .

A vector in fock space of reservoir is a linear combination of basis vectors

$$|N(\vec{k}_1), N(\vec{k}_2), \dots\rangle_B = \frac{(b_{\vec{k}_1})^{N(\vec{k}_1)} (b_{\vec{k}_2})^{N(\vec{k}_2)} \dots}{\sqrt{N(\vec{k}_1)! N(\vec{k}_2)! \dots}} |0\rangle_B \tag{29}$$

where are eigenstates of  $H_B$  and the operators  $b_{\vec{k}}$  and  $b_{\vec{k}}^\dagger$  act on them as

$$\begin{aligned}
b_{\vec{k}} |N(\vec{k}_1), N(\vec{k}_2), \dots, N(\vec{k}), \dots\rangle_B &= \sqrt{N(\vec{k})} |N(\vec{k}_1), N(\vec{k}_2), \dots, N(\vec{k}) - 1, \dots\rangle_B \\
b_{\vec{k}}^\dagger |N(\vec{k}_1), N(\vec{k}_2), \dots, N(\vec{k}), \dots\rangle_B &= \sqrt{N(\vec{k}) + 1} |N(\vec{k}_1), N(\vec{k}_2), \dots, N(\vec{k}) + 1, \dots\rangle_B
\end{aligned} \tag{30}$$

If the state of system in  $t = 0$  is taken to be  $|\psi(0)\rangle = |0\rangle_B \otimes |n_1, n_2, n_3\rangle_\omega$  where  $|0\rangle_B$  is the vacuum state of the reservoir and  $|n_1, n_2, n_3\rangle_\omega$  is an excited state of the Hamiltonian  $H_S = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 \vec{x}^2$ , then it is clear that

$$\langle \psi(0) | H_s(0) | \psi(0) \rangle = (n_1 + n_2 + n_3 + \frac{3}{2})\omega. \tag{31}$$

On the other hand from (27), (28) and (16) we find

$$\begin{aligned}
& \lim_{t \rightarrow \infty} [\langle \psi(0) | : \frac{1}{2} m \dot{\vec{x}}^2 + \frac{1}{2} m \omega^2 \vec{x}^2 : | \psi(0) \rangle] = 0, \\
& \lim_{t \rightarrow \infty} [\langle \psi(0) | : H_s(t) : | \psi(0) \rangle] \\
&= \frac{\beta^2 \omega^4}{2\pi^2 m} \lim_{t \rightarrow \infty} \left| \int_{-\infty}^{+\infty} \frac{dx}{x} \frac{e^{ixt}}{\omega^2 - x^2 + i \frac{\beta}{m} x} \right|^2 \langle n_1, n_2, n_3 | \vec{x}^2(0) | n_1, n_2, n_3 \rangle \\
&\simeq \frac{\beta^2}{2m} \langle n_1, n_2, n_3 | \vec{x}^2(0) | n_1, n_2, n_3 \rangle = \frac{\beta^2}{2m^2 \omega} (n_1 + n_2 + n_3 + \frac{3}{2}), \quad (32)
\end{aligned}$$

where : : denotes the normal ordering operator. Now by substituting  $b_{\vec{k}}(t)$  from (28) into (14), it is easy to show that

$$\begin{aligned}
& \lim_{t \rightarrow \infty} [\langle \psi(0) | : H_B(t) : | \psi(0) \rangle] \\
&= \frac{\beta \omega^4}{\pi} \int_0^\infty \frac{dx}{(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2} \langle n_1, n_2, n_3 | \vec{x}^2(0) | n_1, n_2, n_3 \rangle + \\
&+ \frac{\beta}{\pi m^2} \int_0^\infty \frac{x^2 dx}{(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2} \langle n_1, n_2, n_3 | \vec{p}^2(0) | n_1, n_2, n_3 \rangle \\
&= \frac{\beta \omega^3}{\pi m} (n_1 + n_2 + n_3 + \frac{3}{2}) \int_0^\infty \frac{dx}{(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2} + \\
&+ \frac{\beta \omega}{\pi m} (n_1 + n_2 + n_3 + \frac{3}{2}) \int_0^\infty \frac{x^2 dx}{(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2}. \quad (33)
\end{aligned}$$

For sufficiently weak damping that is when  $\beta$  is very small, the integrands in (33) have singularity points  $x = \pm(\omega_1 \pm \frac{i\beta}{2m})$  and by using residual calculus we find

$$\lim_{t \rightarrow \infty} [\langle \psi(0) | : H_B : | \psi(0) \rangle] = (n_1 + n_2 + n_3 + \frac{3}{2})\omega, \quad (34)$$

Comparing (31) and (34), show that the total energy of oscillator has been transmitted to the reservoir and according to (32), the kinetic energy of oscillator tends to zero.

If the state of system in  $t = 0$  is  $\rho(0) = \rho_B^T \otimes |S\rangle_\omega$  where  $\rho_B^T = \frac{e^{-\frac{H_B}{KT}}}{Tr_B(e^{-\frac{H_B}{KT}})}$  is the Maxwell-Boltzman distribution and  $|S\rangle_\omega$  is an arbitrary state of harmonic oscillator, then by making use of  $Tr_B[b_{\vec{k}}^\dagger(0)b_{\vec{k}'}(0)\rho_B^T] = \frac{\delta(\vec{k}-\vec{k}')}{e^{\frac{\omega_{\vec{k}}}{KT}} - 1}$  one can



show that

$$\begin{aligned} \lim_{t \rightarrow \infty} \langle : \frac{1}{2} m \dot{\vec{x}}^2(t) + \frac{1}{2} m \omega^2 \vec{x}^2(t) : \rangle &= \frac{6\beta}{\pi m^2} \int_0^\infty \frac{x}{[(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2] (e^{\frac{x}{KT}} - 1)} dx \\ &+ \frac{6\beta}{\pi m^2} \int_0^\infty \frac{x^3}{[(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2] (e^{\frac{x}{KT}} - 1)} dx. \end{aligned} \quad (35)$$

### 3.2 Transition probabilities

We can write the Hamiltonian (12) as

$$\begin{aligned} H &= H_0 + H', \\ H_0 &= \sum_{j=1}^3 (a_j^\dagger a_j + \frac{3}{2}) \omega + H_B, \\ H' &= -\frac{\vec{p}}{m} \cdot \vec{R} + \frac{\vec{R}^2}{2m}, \end{aligned} \quad (36)$$

where  $a_j, a_j^\dagger$   $j = 1, 2, 3$  are annihilation and creation operators of the harmonic oscillator. In interaction picture we can write

$$\begin{aligned} a_{jI}(t) &= e^{iH_0 t} a_j(0) e^{-iH_0 t} = a_j(0) e^{-i\omega t} \quad j = 1, 2, 3, \\ b_{\vec{k}I}(t) &= e^{iH_0 t} b_{\vec{k}}(0) e^{-iH_0 t} = b_{\vec{k}}(0) e^{-i\omega_{\vec{k}} t}, \end{aligned} \quad (37)$$

the terms  $\frac{\vec{R}}{m} \cdot \vec{p}$  and  $\frac{\vec{R}^2}{2m}$  are of the first order and second order of damping respectively, therefore, for a sufficiently weak damping,  $\frac{\vec{R}^2}{2m}$  is small in comparison with  $\frac{\vec{R}}{m} \cdot \vec{p}$ . Furthermore  $\frac{\vec{R}^2}{2m}$  has not any role in those transition probabilities where initial and final states of harmonic oscillator are different, hence we can neglect it in  $H'$ . Substituting  $a_{jI}$   $j = 1, 2, 3$  and  $b_{\vec{k}I}$  from (37) in  $\frac{\vec{R}}{m} \cdot \vec{p}$ , one can obtain  $H'_I$  in interaction picture as

$$\begin{aligned} H'_I &= -i \sqrt{\frac{\omega}{2m}} \sum_{j=1}^3 \int_{-\infty}^{+\infty} d^3 k [f(\omega_{\vec{k}}) a_j^\dagger b_{\vec{k}}(0) e^{i(\omega - \omega_{\vec{k}})t} - f^*(\omega_{\vec{k}}) a_j b_{\vec{k}}^\dagger(0) e^{-i(\omega - \omega_{\vec{k}})t} \\ &- f(\omega_k) a_j b_{\vec{k}}(0) e^{-i(\omega_{\vec{k}} + \omega)t} + f^*(\omega_k) a_j^\dagger b_{\vec{k}}^\dagger(0) e^{i(\omega_{\vec{k}} + \omega)t}] k_j, \end{aligned} \quad (38)$$

where  $k_j$   $j = 1, 2, 3$  are the cartesian component of the vector  $\vec{k}$ . The terms containing just  $a_j b_{\vec{k}}(0)$  and  $a_j^\dagger b_{\vec{k}}^\dagger(0)$  violate the conservation of energy in the first order perturbation, because  $a_j b_{\vec{k}}(0)$  destroys an excited state of harmonic

oscillator while at the same time destroying a reservoir excitation state and  $a_j^\dagger b_k^\dagger(0)$  creates an excited state of harmonic oscillator, while creating an excited reservoir state at the same time, therefore we neglect them due to energy conservation and write  $H'_I$  as

$$H'_I = -i\sqrt{\frac{\omega}{2m}} \sum_{j=1}^3 \int_{-\infty}^{+\infty} d^3k [f(\omega_{\vec{k}}) a_j^\dagger b_{\vec{k}}^\dagger(0) e^{i(\omega-\omega_{\vec{k}})t} - f^*(\omega_{\vec{k}}) a_j b_{\vec{k}}^\dagger(0) e^{-i(\omega-\omega_{\vec{k}})t}] k_j. \quad (39)$$

The time evolution of density operator in interaction picture is [14]

$$\rho_I(t) = U_I(t, t_0) \rho_I(t_0) U_I^\dagger(t, t_0), \quad (40)$$

where  $U_I$  is the time evolution operator, which in first order perturbation is

$$\begin{aligned} U_I(t, t_0 = 0) &= 1 - i \int_0^t dt_1 H'_I(t_1) = \\ &= 1 - \sqrt{\frac{\omega}{2m}} \sum_{j=1}^3 \int_{-\infty}^{+\infty} d^3k k_j [f(\omega_{\vec{k}}) a_j^\dagger b_{\vec{k}}^\dagger(0) e^{\frac{i(\omega-\omega_{\vec{k}})t}{2}} \\ &\quad - f^*(\omega_{\vec{k}}) a_j b_{\vec{k}}^\dagger(0) e^{\frac{-i(\omega-\omega_{\vec{k}})t}{2}}] \frac{\sin \frac{(\omega-\omega_{\vec{k}})t}{2}}{\frac{(\omega-\omega_{\vec{k}})}{2}}. \end{aligned} \quad (41)$$

Let  $\rho_I(0) = |n_1, n_2, n_3\rangle_\omega {}_\omega\langle n_1, n_2, n_3| \otimes |0\rangle_B {}_B\langle 0|$  where  $|0\rangle_B$  is the vacuum state of the reservoir and  $|n_1, n_2, n_3\rangle_\omega$ , is an excited state of the harmonic oscillator, substituting  $U_I(t, 0)$  from (41) in (40) and tracing out the reservoir parameters, we obtain the probability that the reservoir absorbs energy  $\omega$  from oscillator

$$\begin{aligned} \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{1}{2})\omega} &= Tr_s[(|n_1-1, n_2, n_3\rangle_\omega {}_\omega\langle n_1-1, n_2, n_3| + \\ &\quad + |n_1, n_2-1, n_3\rangle_\omega {}_\omega\langle n_1, n_2-1, n_3| + |n_1, n_2, n_3-1\rangle_\omega {}_\omega\langle n_1, n_2, n_3-1|) \rho_{sI}(t)] = \\ &= \frac{2\pi\omega(n_1+n_2+n_3)}{3m} \int_0^{+\infty} \omega_{\vec{k}}^4 |f(\omega_{\vec{k}})|^2 \frac{\sin^2 \frac{(\omega_{\vec{k}}-\omega)t}{2}}{(\frac{\omega_{\vec{k}}-\omega}{2})^2}, \end{aligned} \quad (42)$$

where  $Tr_s$  denotes taking trace over harmonic oscillator and  $\rho_{sI}(t)$  is obtained by taking trace of  $\rho_I(t)$  over reservoir parameters i.e  $\rho_{sI}(t) = Tr_B(\rho_I(t))$  and we have used the formula  $Tr_B[|1_{\vec{k}}\rangle_B {}_B\langle 1_{\vec{k}'}|] = \delta(\vec{k} - \vec{k}')$ .

For very large times, we can write  $\frac{\sin^2 \frac{(\omega_{\vec{k}}-\omega)t}{2}}{(\frac{\omega_{\vec{k}}-\omega}{2})^2} = 2\pi t \delta(\omega_{\vec{k}} - \omega)$  which leads to

$$\Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{1}{2})\omega} = \frac{4\pi^2\omega^5(n_1+n_2+n_3)t|f(\omega)|^2}{3m}. \quad (43)$$

For the special choice (22), above transition probability becomes

$$\Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{1}{2})\omega} = \frac{(n_1+n_2+n_3)\beta t}{m}, \quad (44)$$

in this case the oscillator can only absorb energy from reservoir.

Now consider the case where the reservoir is an excited state in  $t = 0$  for example  $\rho_I(0) = |n_1, n_2, n_3\rangle_\omega \langle n_1, n_2, n_3| \otimes |1_{\vec{p}_1}, \dots, 1_{\vec{p}_r}\rangle_B \langle 1_{\vec{p}_1}, \dots, 1_{\vec{p}_r}|$  where  $|1_{\vec{p}_1}, \dots, 1_{\vec{p}_r}\rangle_B$  denotes a state of reservoir that contains  $r$  quanta with corresponding momenta  $\vec{p}_1, \dots, \vec{p}_r$ , then

$$\begin{aligned} Tr_B[b_{\vec{k}}^\dagger |1_{\vec{p}_1}, \dots, 1_{\vec{p}_r}\rangle_B \langle 1_{\vec{p}_1}, \dots, 1_{\vec{p}_r}| b_{\vec{k}'}] &= \delta(\vec{k} - \vec{k}'), \\ Tr_B[b_{\vec{k}} |1_{\vec{p}_1}, \dots, 1_{\vec{p}_r}\rangle_B \langle 1_{\vec{p}_1}, \dots, 1_{\vec{p}_r}| b_{\vec{k}'}^\dagger] &= \sum_{l=1}^r \delta(\vec{k} - \vec{p}_l) \delta(\vec{k}' - \vec{p}_l), \end{aligned} \quad (45)$$

and from the long time approximation, we find

$$\begin{aligned} \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{1}{2})\omega} &= \frac{4\pi^2\omega^5(n_1+n_2+n_3)t}{3m} |f(\omega)|^2, \\ \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{5}{2})\omega} &= \\ \frac{\omega\pi t}{m} |f(\omega)|^2 \sum_{l=1}^r \delta(\omega_{\vec{p}_l} - \omega) [(n_1+1)p_{l1}^2 + (n_2+1)p_{l2}^2 + (n_3+1)p_{l3}^2], \end{aligned} \quad (46)$$

where  $p_{l1}, p_{l2}, p_{l3}$  are the cartesian components of vector  $\vec{p}_l$ . Specially for the choice (22), we have

$$\begin{aligned} \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{1}{2})\omega} &= \frac{(n_1+n_2+n_3)\beta t}{m}, \\ \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{5}{2})\omega} &= \\ = \frac{3\beta t}{4\pi m\omega^4} \sum_{l=1}^r \delta(\omega_{\vec{p}_l} - \omega) [(n_1+1)p_{l1}^2 + (n_2+1)p_{l2}^2 + (n_3+1)p_{l3}^2] \end{aligned} \quad (47)$$

Another important case is when the reservoir has a Maxwell-Boltzman distribution, so let  $\rho_I(0) = |n_1, n_2, n_3\rangle_\omega \langle n_1, n_2, n_3| \otimes \rho_B^T$  where

$\rho_B^T = \frac{e^{\frac{-H_B}{KT}}}{Tr_B(e^{\frac{-H_B}{KT}})}$ , then using the relations

$$\begin{aligned} Tr_B[b_{\vec{k}}\rho_B^T b_{\vec{k}'}] &= Tr_B[b_{\vec{k}}^\dagger \rho_B^T b_{\vec{k}'}^\dagger] = 0, \\ Tr_B[b_{\vec{k}}\rho_B^T b_{\vec{k}'}^\dagger] &= \frac{\delta(\vec{k} - \vec{k}')}{e^{\frac{\omega_{\vec{k}}}{KT}} - 1}, \\ Tr_B[b_{\vec{k}}^\dagger \rho_B^T b_{\vec{k}'}] &= \frac{\delta(\vec{k} - \vec{k}')e^{\frac{\omega_{\vec{k}}}{KT}}}{e^{\frac{\omega_{\vec{k}}}{KT}} - 1}, \end{aligned} \quad (48)$$

we can obtain the following transition probabilities in very long time

$$\begin{aligned} \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{1}{2})\omega} &= \frac{4\pi^2\omega^5(n_1+n_2+n_3)t}{3m} \frac{|f(\omega)|^2 e^{\frac{\omega}{KT}}}{e^{\frac{\omega}{KT}} - 1}, \\ \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{5}{2})\omega} &= \\ &= \frac{4\pi^2\omega^5(n_1+n_2+n_3+3)t}{3m} \frac{|f(\omega)|^2}{e^{\frac{\omega}{KT}} - 1}, \end{aligned} \quad (49)$$

substituting (22) in these recent relations we find

$$\begin{aligned} \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{1}{2})\omega} &= \frac{(n_1+n_2+n_3)\beta t e^{\frac{\omega}{KT}}}{m(e^{\frac{\omega}{KT}} - 1)}, \\ \Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{5}{2})\omega} &= \frac{(n_1+n_2+n_3+3)\beta t}{m(e^{\frac{\omega}{KT}} - 1)}. \end{aligned} \quad (50)$$

So in very low temperatures the energy flows from oscillator to the reservoir by the rate  $\Gamma_{(n_1+n_2+n_3+\frac{3}{2})\omega \rightarrow (n_1+n_2+n_3+\frac{1}{2})\omega} = \frac{(n_1+n_2+n_3)\beta}{m}$ , and no energy flows from the reservoir to the oscillator.

## 4 A dissipative two level quantum system

Let us write the Hamiltonian (12) as

$$H = H_s + H_B - \frac{\vec{R}}{m} \cdot \vec{p} + \frac{\vec{R}^2}{2m}, \quad (51)$$

where  $H_S = \frac{\vec{p}^2}{2m} + v(\vec{x})$  is the Hamiltonian of the system. The  $H_s$  has two eigenvalues  $E_1, E_2$  corresponding to eigenkets  $|1\rangle, |2\rangle$  respectively, and we can write

$$H_s = \frac{1}{2}(E_2 - E_1)\sigma_z + \frac{1}{2}(E_1 + E_2) \quad \sigma_z = |2\rangle\langle 2| - |1\rangle\langle 1|. \quad (52)$$

For sufficiently weak damping we can neglect the term  $\frac{\vec{R}^2}{2m}$ , because it is proportional to the second order of damping and write  $H$  as

$$\begin{aligned} H &= \frac{1}{2}\omega_0\sigma_z + H_B - \int d^3k [(G_{12}\sigma + G_{21}\sigma^\dagger)b_{\vec{k}} - \int d^3kb_{\vec{k}}^\dagger [(G_{21}^*\sigma + G_{12}^*\sigma^\dagger] \\ G_{ij} &= \frac{f(\omega_{\vec{k}})}{m} \vec{k} \cdot \vec{p}_{ij} = i\omega_{ij} f(\omega_{\vec{k}}) \vec{k} \cdot \vec{x}_{ij} \\ \vec{x}_{ij} &= \langle i|\vec{x}|j\rangle \quad \vec{p}_{ij} = \langle i|\vec{p}|j\rangle \quad i, j = 1, 2, \end{aligned} \quad (53)$$

where  $\omega_0 = E_2 - E_1$  and  $\sigma = |1\rangle\langle 2|$ . Using commutation relations

$$[\sigma, \sigma^\dagger] = -\sigma_z \quad [\sigma, \sigma_z] = 2\sigma \quad [\sigma_z, \sigma^\dagger] = 2\sigma^\dagger, \quad (54)$$

one can easily obtain Heisenberg equations for the two level system

$$\begin{aligned} \dot{\sigma} &= i[H, \sigma] = -i\omega_0\sigma - i \int d^3k [G_{21}\sigma_z b_{\vec{k}} + G_{12}^* b_{\vec{k}}^\dagger \sigma_z] \\ \dot{\sigma}_z &= i[H, \sigma_z] = -2i \int d^3k [G_{12}\sigma b_{\vec{k}} - G_{21}\sigma^\dagger b_{\vec{k}}] - 2i \int d^3k [G_{21}^* b_{\vec{k}}^\dagger \sigma - G_{12}^* b_{\vec{k}}^\dagger \sigma^\dagger]. \end{aligned} \quad (55)$$

Since equal-time system and reservoir operators commute, we can write the Heisenberg equations (55) in different but equivalent ways. For example we can use the normal ordering where annihilation operator of the reservoir  $b_{\vec{k}}$  appears at the right and creation operator  $b_{\vec{k}}^\dagger$  appears at the left of the system operators, i.e,  $\sigma, \sigma^\dagger, \sigma_z$ , or we can use antinormal ordering where annihilation operator of the reservoir  $b_{\vec{k}}$  appears at the left and creation operator  $b_{\vec{k}}^\dagger$  appears at the right of the system operators, i.e,  $\sigma, \sigma^\dagger, \sigma_z$ . In equations (55) we have used the normal ordering.

One can easily obtain the Heisenberg equation for  $b_{\vec{k}}$  as

$$\dot{b}_{\vec{k}} = i[H, b_{\vec{k}}] = -i\omega_{\vec{k}} b_{\vec{k}} + i \int d^3k G_{21}^* \sigma + i \int d^3k G_{12}^* \sigma^\dagger, \quad (56)$$

with the following formal solution

$$b_{\vec{k}}(t) = b_{\vec{k}}(0)e^{-i\omega_{\vec{k}}t} + iG_{21}^* \int_0^t dt' \sigma(t')e^{-i\omega_{\vec{k}}(t-t')} + iG_{12}^* \int_0^t dt' \sigma^\dagger(t')e^{-i\omega_{\vec{k}}(t-t')}. \quad (57)$$

Let us assume that damping is sufficiently weak that we can take the itegrands (57) we can take

$$\sigma(t') \cong \sigma(t)e^{-i\omega_0(t'-t)} \quad \sigma^\dagger(t') \cong \sigma^\dagger(t)e^{i\omega_0(t'-t)}. \quad (58)$$

This is called the Markovian approximation [15] which replaces the system operators in (57) by an operator that depends on the system operators at the same time  $t$ , without taking into account the memory of these operators at earlier times. Substituting (58) in (57) and using (55) we find

$$\begin{aligned} \langle \dot{\sigma} \rangle &= -i\omega_0 \langle \sigma \rangle + \int d^3k |G_{21}|^2 [\langle \sigma_z \sigma \rangle \int_0^t dt' e^{i(\omega_{\vec{k}} - \omega_0)(t'-t)} - \langle \sigma \sigma_z \rangle \int_0^t dt' e^{i(\omega_{\vec{k}} + \omega_0)(t'-t)}] \\ &+ \int d^3k G_{21} G_{12}^* [\langle \sigma_z \sigma^\dagger \rangle \int_0^t dt' e^{i(\omega_{\vec{k}} + \omega_0)(t'-t)} - \langle \sigma^\dagger \sigma_z \rangle \int_0^t dt' e^{i(\omega_{\vec{k}} - \omega_0)(t'-t)}] \\ \langle \dot{\sigma}_z \rangle &= 2\langle \sigma \sigma^\dagger \rangle \int d^3k |G_{12}|^2 [\int_0^t dt' e^{i(\omega_{\vec{k}} + \omega_0)(t'-t)} + \int_0^t dt' e^{-i(\omega_{\vec{k}} + \omega_0)(t'-t)}] \\ &- 2\langle \sigma^\dagger \sigma \rangle \int d^3k |G_{21}|^2 [\int_0^t dt' e^{i(\omega_{\vec{k}} - \omega_0)(t'-t)} + \int_0^t dt' e^{-i(\omega_{\vec{k}} - \omega_0)(t'-t)}], \end{aligned} \quad (59)$$

where we have taken expectation values in state  $|\psi(0)\rangle = |0\rangle_B \otimes |S\rangle$  which is a tensor product of the vacuum state of the reservoir  $|0\rangle_B$  and an arbitray

state of the system,  $|S\rangle$ . In long time approximation, we can write

$$\begin{aligned}
& \int d^3k |G_{21}|^2 \int_0^t dt' e^{i(\omega_{\vec{k}} - \omega_0)(t' - t)} = \\
& = \frac{4\pi\omega_0^2 |\vec{x}_{21}|^2}{3} \int_0^\infty d\omega_{\vec{k}} |f(\omega_{\vec{k}})|^2 \omega_{\vec{k}}^4 \left[ \frac{-i}{\omega_{\vec{k}} - \omega_0} + \pi\delta(\omega_{\vec{k}} - \omega_0) \right] \\
& \int d^3k |G_{21}|^2 \int_0^t dt' e^{i(\omega_{\vec{k}} + \omega_0)(t' - t)} = -i \frac{4\pi\omega_0^2 |\vec{x}_{21}|^2}{3} \int_0^\infty d\omega_{\vec{k}} \frac{|f(\omega_{\vec{k}})|^2 \omega_{\vec{k}}^4}{\omega_{\vec{k}} + \omega_0} \\
& \int d^3k G_{21} G_{12}^* \int_0^t dt' e^{i(\omega_{\vec{k}} - \omega_0)(t' - t)} = \\
& = -\frac{4\pi\omega_0^2 |\vec{x}_{21}|^2}{3} \int_0^\infty d\omega_{\vec{k}} |f(\omega_{\vec{k}})|^2 \omega_{\vec{k}}^4 \left[ \frac{-i}{\omega_{\vec{k}} - \omega_0} + \pi\delta(\omega_{\vec{k}} - \omega_0) \right] \\
& \int d^3k G_{21} G_{12}^* \int_0^t dt' e^{i(\omega_{\vec{k}} + \omega_0)(t' - t)} = i \frac{4\pi\omega_0^2 |\vec{x}_{21}|^2}{3} \int_0^\infty d\omega_{\vec{k}} \frac{|f(\omega_{\vec{k}})|^2 \omega_{\vec{k}}^4}{\omega_{\vec{k}} + \omega_0},
\end{aligned} \tag{60}$$

substitution of ( 60) in (59) gives

$$\begin{aligned}
\langle \dot{\sigma}_z \rangle &= -4\mu \langle \sigma^\dagger \sigma \rangle = -2\mu(1 + \langle \sigma_z \rangle) & \mu &= \frac{4\pi^2 \omega_0^6 |f(\omega_0)|^2}{3} \\
\langle \dot{\sigma} \rangle &= -i\omega_0 \langle \sigma \rangle + (i\Delta_1 - i\Delta_2 - \mu) \langle \sigma \rangle + (i\Delta_2 - i\Delta_1 - \mu) \langle \sigma^\dagger \rangle \\
\Delta_1 &= \frac{4\pi^2 \omega_0^6 |\vec{x}_{12}|^2}{3} \int_0^\infty d\omega_{\vec{k}} \frac{|f(\omega_{\vec{k}})|^2 \omega_{\vec{k}}^4}{\omega_{\vec{k}} - \omega_0} & \Delta_2 &= \frac{4\pi^2 \omega_0^6 |\vec{x}_{12}|^2}{3} \int_0^\infty d\omega_{\vec{k}} \frac{|f(\omega_{\vec{k}})|^2 \omega_{\vec{k}}^4}{\omega_{\vec{k}} + \omega_0}.
\end{aligned} \tag{61}$$

The solution of the first equation in (61) is

$$\langle \sigma_z(t) \rangle = -1 + (1 + \langle \sigma_z(0) \rangle) e^{-2\mu t} \tag{62}$$

If  $\langle \sigma_z(0) \rangle = 1$ , i.e., the two level system is initially in the upper state  $|2\rangle$ , it decays to the lower state  $\langle \sigma_z \rangle = -1$  with the rate  $2\mu$ , and if the system is initially in the lower state, it remains for all times in that state.

By defining  $\hat{F} = \sigma + \sigma^\dagger$  and  $\hat{E} = \sigma^\dagger - \sigma$ , the second equation in (61) can be written as

$$\begin{aligned}
\langle \dot{\hat{F}} \rangle &= i\Gamma \langle \hat{E} \rangle - 2\mu \langle \hat{F} \rangle & \Gamma &= \omega_0 - 2\Delta_2 - 2\Delta_1, \\
\langle \dot{\hat{E}} \rangle &= i\omega_0 \langle \hat{F} \rangle,
\end{aligned} \tag{63}$$

with the following solutions

$$\begin{aligned}\langle \hat{F} \rangle &= \hat{C}_1 e^{i\Omega_+ t} + \hat{C}_2 e^{i\Omega_- t}, & \Omega_{\pm} &= i\mu \pm i\sqrt{\mu^2 + \omega_0 \Gamma} \\ \langle \hat{E} \rangle &= \frac{\omega_0}{\Omega_+} \hat{C}_1 e^{i\Omega_+ t} + \frac{\omega_0}{\Omega_-} \hat{C}_2 e^{i\Omega_- t}.\end{aligned}\quad (64)$$

If  $\mu^2 + \omega_0 \Gamma < 0$ , we conclude that  $\langle \sigma \rangle$  decays with the rate  $\mu$ .

## 5 Quantum field of the reservoir

Let us define the operators  $Y(\vec{x}, t)$  and  $\Pi_Y(\vec{x}, t)$  as follows

$$\begin{aligned}Y(\vec{x}, t) &= \int_{-\infty}^{+\infty} \frac{d^3 k}{\sqrt{2(2\pi)^3 \omega_{\vec{k}}}} (b_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}} + b_{\vec{k}}^\dagger(t) e^{-i\vec{k} \cdot \vec{x}}), \\ \Pi_Y(\vec{x}, t) &= i \int_{-\infty}^{+\infty} d^3 k \sqrt{\frac{\omega_{\vec{k}}}{2(2\pi)^3}} (b_{\vec{k}}^\dagger(t) e^{-i\vec{k} \cdot \vec{x}} - b_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}}),\end{aligned}\quad (65)$$

then using commutation relations (15), one can show that  $Y(\vec{x}, t)$  and  $\Pi_Y(\vec{x}, t)$ , satisfy the equal time commutation relations

$$[Y(\vec{x}, t), \Pi_Y(\vec{x}', t)] = i\delta(\vec{x} - \vec{x}'), \quad (66)$$

furthermore by substituting  $b_{\vec{k}}(t)$  from (20) in (65), we obtain

$$\begin{aligned}\frac{\partial \Pi_Y(\vec{x}, t)}{\partial t} &= \nabla^2 Y + 2\dot{\vec{x}}(t) \cdot \vec{M}(\vec{x}), & \vec{M}(\vec{x}) &= Re \int_{-\infty}^{+\infty} d^3 k \sqrt{\frac{\omega_{\vec{k}}}{2(2\pi)^3}} f(\omega_{\vec{k}}) \vec{k} e^{-i\vec{k} \cdot \vec{x}}, \\ \Pi_Y(\vec{x}, t) &= \frac{\partial Y}{\partial t} - 2\dot{\vec{x}}(t) \cdot \vec{N}(\vec{x}), & \vec{N}(\vec{x}) &= Im \int_{-\infty}^{+\infty} d^3 k \frac{f(\omega_{\vec{k}})}{\sqrt{2(2\pi)^3 \omega_{\vec{k}}}} \vec{k} e^{-i\vec{k} \cdot \vec{x}},\end{aligned}\quad (67)$$

so  $Y(\vec{x}, t)$  satisfies the following source included Klein-Gordon equation

$$\frac{\partial^2 Y}{\partial t^2} - \nabla^2 Y = 2\ddot{\vec{x}}(t) \cdot \vec{N}(\vec{x}) + 2\dot{\vec{x}}(t) \cdot \vec{M}(\vec{x}), \quad (68)$$

with the corresponding Lagrangian density

$$\mathcal{L} = \frac{1}{2} \left( \frac{\partial Y}{\partial t} \right)^2 - \frac{1}{2} \nabla Y \cdot \nabla Y - 2\dot{\vec{x}} \cdot \vec{N}(\vec{x}) \frac{\partial Y}{\partial t} + 2\dot{\vec{x}} \cdot \vec{M}(\vec{x}) Y. \quad (69)$$



Therefore the reservoir is a massless Klein-Gordon field with source  $2\ddot{\vec{x}}.\vec{N}(\vec{x})+2\dot{\vec{x}}.\vec{M}(\vec{x})$ . The Hamiltonian density for (68) is as follows

$$\mathfrak{H} = \frac{(\Pi_Y + 2\dot{\vec{x}}.\vec{N})^2}{2} + \frac{1}{2}|\nabla\vec{Y}|^2 - 2\dot{\vec{x}}.\vec{M}Y, \quad (70)$$

and equations (67) are Heisenberg equations for  $Y$  and  $\Pi_Y$ . If we obtain  $b_{\vec{k}}$  and  $b_{\vec{k}}^\dagger$  from (65) in terms of  $Y$  and  $\Pi_Y$  and substitute them in  $H_B$  defined in (14), we find

$$H_B = \int_{-\infty}^{+\infty} d^3k \omega_{\vec{k}} b_{\vec{k}}^\dagger b_{\vec{k}} = \frac{\Pi_Y^2}{2} + \frac{1}{2}|\nabla\vec{Y}|^2. \quad (71)$$

## 6 Concluding remarks

By generalizing Caldeira-Legget model to a reservoir with continuous degrees of freedom, for example a Klein-Gordon field, a new minimal coupling method introduced which can be extended and applied to a large class of dissipative quantum systems consistently. This method applied to an isotropic three dimensional quantum damped harmonic oscillator and a dissipative two level system as prototypes of important dissipative models. Some transition probabilities explaining the way energy flows between subsystems obtained. Choosing different coupling functions in (16) we could investigate another classes of dissipative systems.

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